Q No→ If $u - v = (x-y)(x^2+4xy+y^2)$ and $f(z) = u+iv$ is an analytic function of $z = x+iy$, find $f(z)$ in terms of $z$.

Soln. We have,

$f(z) = u + iv$ So that $if(z) = iu - v$

$\therefore (1+i)f(z) = (u-v) + i(u+v)$

$= U + iV$ where $U = u - v$, $V = u + v$.

Here, $U = u - v = (x-y)(x^2+4xy+y^2)$

$$\therefore \frac{\partial U}{\partial x} = \frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} = (x^2+4xy+y^2) + (x-y)(2x+4y) = \phi_1(x,y)$$

$$= x^2+4xy+y^2+2x^2+4xy-2xy+4y^2$$

$$= 3x^2+6xy-3y^2.$$

and
$$\frac{\partial U}{\partial y} = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} = -(x^2+4xy+y^2) + (x-y)(4x+2y)$$

$$= 3x^2 - 6xy - 3y^2$$

Let $\frac{\partial U}{\partial x} = \phi_1(x,y)$, and $\frac{\partial U}{\partial y} = \phi_2(x,y)$

then $(1+i)f(z) = \int [\phi_1(z,0) - i\phi_2(z,0)]\,dz + C$

$$= \int (3z^2 - 3iz^2)\,dz + C = (1-i)z^3 + C.$$

$$\therefore f(z) = \frac{(1-i)}{(1+i)} z^3 + C = \frac{(1-i)^2}{(1+i)(1-i)} z^3 + C$$

$$= \frac{1-2i+i^2}{1+1} z^3 + C = -iz^3 + C.$$

QN.⇒ If $u+v = \frac{2\sin 2x}{e^{2y} - e^{-2y} - 2\cos 2x}$, find the analytic function $f(z) = u+iv$.

Solun:- We have, $(1+i)f(z) = (u-v) + i(u+v)$

Let $u - v = U$ and $u+v = V$, So that, $(1+i)f(z) = U + iV$.

$$\therefore (1+i)f'(z) = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x} = \frac{\partial V}{\partial y} + i\frac{\partial V}{\partial x}$$

by Cauchy-Riemann equations.

Now, $$\frac{\partial V}{\partial y} = \frac{-2\sin 2x(2e^{2y} - 2e^{-y})}{(e^{2y}+e^{-2y}-2\cos 2x)^2} = \phi_1(x,y)$$

and $$\frac{\partial V}{\partial x} = 2\left[\frac{2\cos 2x\,(e^{2y}+e^{-2y}-2\cos 2x) - 4\sin^2 2x}{(e^{2y}+e^{-2y}-2\cos 2x)^2}\right]$$

$$= 4\left[\frac{\cos 2x\,(e^{2y}+e^{-2y}) - 2}{(e^{2y}+e^{-2y}-2\cos 2x)^2}\right] = \phi_2(x,y)$$

$\therefore (1+i)f(z) = \int [\phi_1(z,0) + i\phi_2(z,0)]\,dz + c$

$= \int \left[0 + i\dfrac{4(2\cos 2z - 2)}{(2-2\cos 2z)^2}\right] dz + c$

$= 2i\int \dfrac{dz}{\cos 2z - 1} + c = -i\int \operatorname{cosec}^2 z\,dz + c.$

$= i\cot z + c.$

$\therefore f(z) = \dfrac{i}{1+i}\cot z + \dfrac{c}{1+i} = \dfrac{1}{2}(1+i)\cot z + \alpha,$

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Q.No → If $\phi$ and $\psi$ are functions of $x$ & $y$ satisfying Laplace's equation, show that $s + it$ is holomorphic where,

$$s = \frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial x} \text{ and } t = \frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y}.$$

Solu^n It is only necessary to show that $s$ & $t$ satisfy Cauchy-Riemann equation, i.e. we show that,

$$\frac{\partial s}{\partial x} = \frac{\partial t}{\partial y} \text{ and } \frac{\partial s}{\partial y} = -\frac{\partial t}{\partial x}.$$

Since, $\phi$ & $\psi$ satisfy Laplace's equation, we have

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \quad \text{--- (1)}$$

Now, $\dfrac{\partial s}{\partial x} = \dfrac{\partial^2 \phi}{\partial x \partial y} - \dfrac{\partial^2 \psi}{\partial x^2}$ and $\dfrac{\partial t}{\partial y} = \dfrac{\partial^2 \phi}{\partial y \partial x} + \dfrac{\partial^2 \psi}{\partial y^2}$

These give, $\dfrac{\partial t}{\partial y} - \dfrac{\partial s}{\partial x} = \dfrac{\partial^2 \psi}{\partial x^2} + \dfrac{\partial^2 \psi}{\partial y^2} = 0$ by (1)

$$\therefore \frac{\partial s}{\partial x} = \frac{\partial t}{\partial y}.$$

Similarly, $\frac{\partial s}{\partial y} + \frac{\partial t}{\partial x} = \frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 \psi}{\partial y \partial x} + \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \psi}{\partial x \partial y}$

$$= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \text{ (by (1))}$$

Hence, $\frac{\partial s}{\partial y} = -\frac{\partial t}{\partial x}$.

Therefore, $s$ & $t$ satisfy Cauchy-Riemann equations. It follows that $s + it$ is holomorphic

QNo→ Show that the function $e^x(\cos y + i\sin y)$ is holomorphic and find its derivative.

Soln:- Let $f(z) = u + iv = e^x(\cos y + i\sin y)$.

$$= e^x \cos y + i \cdot e^x \sin y$$

We have,

$$\frac{\partial u}{\partial x} = e^x \cos y, \quad \frac{\partial u}{\partial y} = -e^x \sin y,$$

$$\frac{\partial v}{\partial x} = e^x \sin y, \quad \frac{\partial v}{\partial y} = e^x \cos y.$$

These relations show that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \;\&\; \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

So that $u, v$ satisfy Cauchy-Riemann equations.

Also it can be easily shown that $u$ & $v$ are harmonic functions.

Hence, $f'(z)$ is holomorphic.

Now, $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = e^x \cos y + i e^x \sin y$

$$= e^x(\cos y + i\sin y)$$

$$= e^{x+iy} = e^z.$$

Here the derivative is identical with the given function. This result is similar to that of the real function $e^x$.

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Q No → Show that the function $f(z) = \sqrt{|xy|}$ is not analytic at the origin although the Cauchy-Riemann equations are satisfied at that point.

Soln. We have

$$f(z) = u(x,y) + iv(x,y)$$

$$\therefore u(x,y) = \sqrt{|xy|}, \; \& \; v(x,y) = 0.$$

We then have at the origin,

$$\frac{\partial u}{\partial x} = \underset{x\to 0}{Lt.} \frac{u(x,0) - u(0,0)}{x} = \underset{x\to 0}{Lt.} \frac{0-0}{x} = 0.$$

$$\frac{\partial u}{\partial y} = \underset{y\to 0}{Lt.} \frac{u(0,y) - u(0,0)}{y} = \underset{y\to 0}{Lt.} \frac{0-0}{y} = 0.$$

and,

$$\frac{\partial v}{\partial x} = \underset{x\to 0}{Lt.} \frac{v(x,0) - v(0,0)}{x} = \underset{x\to 0}{Lt.} \frac{0-0}{x} = 0.$$

$$\frac{\partial v}{\partial y} = \underset{y\to 0}{Lt.} \frac{v(0,y) - v(0,0)}{y} = \underset{y\to 0}{Lt.} \frac{0-0}{y} = 0.$$

Hence, clearly $u_x = v_y$, & $u_y = -v_y$.

Therefore, Cauchy Riemann equation are satisfied, we have

$$f'(0) = \underset{z\to 0}{Lt.} \frac{f(z) - f(0)}{z} = \underset{z\to 0}{Lt.} \frac{\sqrt{|xy|} - 0}{x+iy} = \underset{z\to 0}{Lt.} \frac{\sqrt{|xy|}}{x+iy}$$

Let $z \to 0$, along ~~the~~ $y = mx$, we get

$$\therefore f'(0) = \underset{x \to 0}{Lt.} \frac{\sqrt{|mx^2|}}{x + imx} = \underset{x \to 0}{Lt.} \frac{\cancel{x}\sqrt{|m|}}{\cancel{x}(1+im)}$$

$$= \frac{\sqrt{|m|}}{1+im}$$

Which depend upon m, Hence it can not be unique.

Therefore, $f'(0)$ does not exist.

Hence, $f(z)$ is not analytic at the origin ($z=0$) although Cauchy Riemann eqn. are satisfied.